A canonical reduction of order for the Kepler problem

1991 J. Phys. A: Math. Gen. 24 5465

(http://iopscience.iop.org/0305-4470/24/23/013)

J. Phys. A: Math. Gen. 24 (1991) 5465–5475. Printed in the UK

A canonical reduction of order for the Kepler problem

S E Godfrey† and G E Prince†

Department of Mathematics, Statistics and Computing Science, University of New England, Armidale, New South Wales 2351, Australia

Received 17 October 1990, in final form 1 July 1991

Abstract. Using a new reduction of order technique we obtain four conserved quantities of the classical planar Kepler problem via the symmetries of the equations of motion and an extra vector field. By passing to a suitable quotient manifold of the evolution space we deduce that negative energy orbits are closed and periodic, without having to solve the differential equation.

1. Introduction

In this paper we look at the classical Kepler problem through the application of a new reduction of order technique [1]. We believe that this technique provides a systematic means of producing first integrals from group actions more general than symmetries without relying on the existence of a variational principle. Moreover, it is a powerful tool for investigating global properties such as periodicity. The Kepler problem is a good vehicle for a demonstration of these features.

The planar equations of motion are defined by a second-order differential equation field on five-dimensional evolution space. There are three point symmetries which take solutions of the equation of motion into other solutions: time translation, which leaves the solution curves invariant but alters their starting times; rotation about the origin, which again leaves the curves invariant but alters the initial angular position; and the transformation associated with the Runge–Lenz vector which allows the orientation and eccentricity of the orbits to remain the same but changes the semi-latus rectum. We wish to show that conserved quantities of the system, i.e. the objects which remain invariant along solution curves, can be obtained using alternative techniques to Noether's theorem [2] or solving the differential equation [3, 4]. Lie's method [2] provides us with the three point symmetries mentioned above. To effect our technique a fourth vector field is required which is a symmetry of the integrable distribution made up of the second-order differential equation field and its three symmetries. To find this field we modify the usual reduction of order via symmetry method by taking successive quotients of the extended phase space by the three actions (for example, we project down to a four-dimensional manifold by setting the time coordinate to a constant, thus identifying all the orbits which only differ by a time translation) and we reduce our differential equation field to first order on a two-dimensional quotient.

† Permanent address: Department of Mathematics, La Trobe University, Bundoora, Victoria 3083, Australia.

Here we obtain a symmetry of the projected vector field which we pull back to the full evolution space, providing us with the fourth vector field. We emphasize that this vector field is not a symmetry of the differential equation but of this equation and its symmetries. We obtain four conserved quantities of the system *without recourse to canonical coordinates* using a theorem presented in [1], which involves constructing (in this case) four 1-forms from the four vector fields. For completeness, we make the conserved quantities our new coordinates for the evolution space and find the basis of vector fields (which includes the second-order differential equation field) dual to the closed 1-forms constucted from them. In this way we find four commuting symmetries for the problem.

The periodicity of the one-dimensional simple harmonic oscillator was investigated by Wulfman and Wybourne [5] through the inspection of a compact subgroup of the symmetry generators which contained the time translation generator. In the case of the planar Kepler problem we found no obvious so(3) subalgebra so we looked for an alternative approach to the proof of periodicity of the negative energy orbits. The two-dimensional quotient described above turns out to be a convenient place to investigate properties of the solution curves. We find that one of the conserved quantities which involves the energy can be expressed entirely by the coordinates of this quotient. Furthermore, the projected differential equation field is tangent to each level curve of this quantity. It turns out that, for negative energy values, these curves are closed. By a simple argument, we deduce that this is also true for the phase-space counterparts of the integral curves of our original differential equation field. Finally, by projecting down to the base manifold, we deduce that the orbits are closed and periodic for negative energy values, which is in agreement with known results on the Kepler problem [3, 4]. Thus we obtain four conserved quantities of the system and deduce the periodicity of the closed solution curves without having to solve the differential equation.

The paper is organized as follows. In section 2 we project the second-order differential equation field for the Kepler problem down onto a two-dimensional quotient. Here we produce a symmetry of the projected field which we identify as our fourth vector field. In section 3 we give an explanation of how a reduction of order is possible using our four vector fields and present the theorem from [1] which we will use to find our four conserved quantities. With these we construct a basis of closed 1-forms and a dual basis of vector fields. In section 4 we obtain a first integral of the projected differential equation field on two-dimensional coordinate submanifolds. By considering the curves on each level surface we identify the range of energy values for which the solution curves are periodic.

2. Symmetries

The classical equation of motion for a particle in a gravitational field moving in $R^3\backslash\{0\}$ is given by

$$\ddot{\boldsymbol{r}} + \frac{\mu \boldsymbol{r}}{r^3} = 0$$

where $\mu > 0$ is the constant field strength. Each orbit is planar and so in any plane with polar coordinates (r, θ) we have the system of equations

$$\ddot{r} = r\dot{\theta}^2 - \frac{\mu}{r^2} \tag{1a}$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} \,. \tag{1b}$$

The three well-known conserved quantities are: the energy $\mathcal{E} := \frac{1}{2}(\dot{r}^2 + h^2/r^2) - \mu/r$; the magnitude of the angular momentum $h := |\boldsymbol{L}| = r^2\dot{\theta}$ and the Runge–Lenz vector $\boldsymbol{R} := \dot{\boldsymbol{r}} \times \boldsymbol{L} - \mu\boldsymbol{r}/r$ (see [2]). The configuration space is now $M = R^2\backslash\{0\}$ and the second-order differential equation field $\Gamma \in \chi(E)$ ($E := R \times TM$ is the evolution space) whose integral curves are lifted orbits of (1*a*) and (1*b*) is

$$\Gamma := \frac{\partial}{\partial t} + \dot{r}\frac{\partial}{\partial r} + \dot{\theta}\frac{\partial}{\partial \theta} + (r\dot{\theta}^2 - \mu/r^2)\frac{\partial}{\partial \dot{r}} - \frac{2\dot{r}\dot{\theta}}{r}\frac{\partial}{\partial \dot{\theta}} \,. \tag{2}$$

Equivalently, Γ is defined by $\langle \Gamma, \mathrm{d}t \rangle = 1$, $\langle \Gamma, \theta^i \rangle = 0$ and $\langle \Gamma, \phi^i \rangle = 0$, where $\theta^1 := \mathrm{d}r - \dot{r}\mathrm{d}t$ and $\theta^2 := \mathrm{d}\theta - \dot{\theta}\mathrm{d}t$ are the *contact forms* and $\phi^1 := \mathrm{d}\dot{r} - \ddot{r}\mathrm{d}t$ and $\phi^2 := \mathrm{d}\dot{\theta} - \ddot{\theta}\mathrm{d}t$ are the *force forms* ($\ddot{r}$ and $\ddot{\theta}$ being substituted from (1*a*) and (1*b*)). There are three one-parameter group actions on $R \times M$ that map graphs of orbits into themselves [2]. These are called *point* or *Lie symmetries* and are generated by the following vector fields on $R \times M$:

$$X_4 := \frac{\partial}{\partial t} \qquad X_3 := \frac{\partial}{\partial \theta} \qquad X_2 := t\frac{\partial}{\partial t} + \frac{2}{3}r\frac{\partial}{\partial r} \,.$$

(The strange ordering here is used to avoid confusion in section 3.) These vector fields form a Lie algebra $[X_4, X_3] = 0$, $[X_3, X_2] = 0$ and $[X_2, X_4] = -X_4$. Any vector field X on $R \times M$, has associated with it a unique vector field $X^{(1)}$ on E called the *(first) prolongation* of X such that $X^{(1)}$ projects onto X and for any contact 1-form α, $\mathcal{L}_{X^{(1)}}\alpha$ is also a contact 1-form [6]. The first prolongations of X_4, X_3, X_2 are

$$X_4^{(1)} = \frac{\partial}{\partial t} \qquad X_3^{(1)} = \frac{\partial}{\partial \theta} \qquad X_2^{(1)} = t\frac{\partial}{\partial t} + \frac{2}{3}r\frac{\partial}{\partial r} - \frac{1}{3}\dot{r}\frac{\partial}{\partial \dot{r}} - \dot{\theta}\frac{\partial}{\partial \dot{\theta}}$$

respectively, with

$$\mathcal{L}_{X_4^{(1)}}\Gamma = 0 \qquad \mathcal{L}_{X_3^{(1)}}\Gamma = 0 \qquad \mathcal{L}_{X_2^{(1)}}\Gamma = -\Gamma$$

where $\mathcal{L}_Y X$ denotes the Lie derivative along $Y \in \chi(E)$ of $X \in \chi(E)$.

There are insufficient symmetries to effect a complete reduction of order of the problem, so we need a fourth field X_1 which is a symmetry of the integral distribution $\mathcal{D} := \mathrm{span}(\{\Gamma, X_4^{(1)}, X_3^{(1)}, X_2^{(1)}\})$, i.e. $\mathcal{L}_{X_1}\mathcal{D} \subseteq \mathcal{D}$. To find this field we take successive quotients by the above actions as in Olver [7]. We can project by the action of a vector field X if and only if $\mathcal{L}_X\Gamma = \lambda X$ for some function λ on E. Considering the above Lie derivatives, we can project by $X_4^{(1)}$ and $X_3^{(1)}$ onto a three-dimensional quotient with coordinates which we can identify as $(r, \dot{r}, \dot{\theta})$. The projected vector fields of Γ and $X_2^{(1)}$ are

$$\tilde{\Gamma} := \dot{r}\frac{\partial}{\partial r} + (r\dot{\theta}^2 - \mu/r^2)\frac{\partial}{\partial \dot{r}} - \frac{2\dot{r}\dot{\theta}}{r}\frac{\partial}{\partial \dot{\theta}}$$

$$\tilde{X}_2 := \frac{2}{3}r\frac{\partial}{\partial r} - \frac{1}{3}\dot{r}\frac{\partial}{\partial \dot{r}} - \dot{\theta}\frac{\partial}{\partial \dot{\theta}}.$$

We are unable to project by $\tilde{X}_2$ since $\mathcal{L}_{\tilde{X}_2}\tilde{\Gamma} = -\tilde{\Gamma}$ and we require $\mathcal{L}_{\tilde{X}_2}\tilde{\Gamma} = \lambda\tilde{X}_2$. Nevertheless, we can produce a multiple of $\tilde{\Gamma}$ which does pass to the quotient. Noting that

$$\tilde{X}_2(\tfrac{3}{2}\log r) = 1 \qquad \tilde{X}_2(\dot{r}r^{1/2}) = 0 \qquad \tilde{X}_2(\dot{\theta}r^{3/2}) = 0$$

we may make a change of coordinates on this quotient

$$r^* = \tfrac{3}{2}\log r \qquad v^1 = \dot{r}r^{1/2} \qquad v^2 = \dot{\theta}r^{3/2}$$

so that

$$\tilde{\Gamma} = \frac{3}{2}\frac{\dot{r}}{r}\left[\frac{\partial}{\partial r^*} + \left(\frac{1}{3}v^1 + \frac{2}{3}\frac{v^{2^2}}{v^1} - \frac{2}{3}\frac{\mu}{v^1}\right)\frac{\partial}{\partial v^1} - \frac{1}{3}v^2\frac{\partial}{\partial v^2}\right].$$

Thus an appropriate multiple of $\tilde{\Gamma}$ is $(2r/\dot{r})\tilde{\Gamma}$ since $\mathcal{L}_{\tilde{X}_2}((2r/\dot{r})\tilde{\Gamma}) = 0$, where $\tilde{X}_2 = \partial/\partial r^*$. We can now project by $\tilde{X}_2$ to a two-dimensional quotient with coordinates which we can identify with (v^1, v^2), where our projected vector field is

$$\hat{\Gamma} = \left(v^1 + 2\frac{v^{2^2}}{v^1} - 2\frac{\mu}{v^1}\right)\frac{\partial}{\partial v^1} - v^2\frac{\partial}{\partial v^2}\ .$$

A symmetry X_1 of $\hat{\Gamma}$ with $\mathcal{L}_{X_1}\hat{\Gamma} = 0$ is

$$X_1 := \frac{1}{v^1 v^{2^2}}\frac{\partial}{\partial v^1}.$$

Pulling the chart back to E as part of the coordinate chart $(t, \theta, r^*, v^1, v^2)$ and converting back to the original coordinates $(t, r, \theta, \dot{r}, \dot{\theta})$ we have, by an abuse of notation,

$$X_1 = \frac{1}{\dot{r}\dot{\theta}^2 r^4}\frac{\partial}{\partial \dot{r}}.$$

Although X_1 is a symmetry of $\hat{\Gamma}$ by construction, it is not a symmetry of Γ since

$$\mathcal{L}_{X_1}\Gamma = -\frac{1}{\dot{r}^2\dot{\theta}^2 r^4}(X_4^{(1)} + \dot{\theta}X_3^{(1)} - \Gamma).$$

However, X_1 is guaranteed to be a symmetry of the integrable distribution $\mathcal{D}$. We note that this is not a global statement because of the singularities of X_1. X_1 could, of course, have been found without resorting to the quotient technique, but this is an effective way of finding it.

3. First integrals

We will briefly digress to explain how the techniques in [1] allow a reduction using our four vector fields. Suppose that f is a first integral of a second-order vector field

$$\Gamma := \frac{\partial}{\partial t} + u^a\frac{\partial}{\partial x^a} + \Lambda^a\frac{\partial}{\partial u^a}$$

on an n-dimensional manifold M (assumed smooth, second-countable, and Hausdorff) where (t, x^a, u^a) are local coordinates for $E = R \times TM$ and Λ^a are smooth functions on E. Then $\mathrm{d}f$ will be a linear combination of the *contact forms* $\theta^a := \mathrm{d}x^a - u^a \mathrm{d}t$ and the *force forms* $\phi^a := \mathrm{d}u^a - \Lambda^a \mathrm{d}t$ since $\Gamma(f) = 0$. A *distribution* $\mathcal{D}$ is an assignment of a vector subspace T_pM to each $p \in M$ and, when regarded as a vector subspace of $\chi(M)$, is *in involution* if $[X, Y] \in \mathcal{D}$ for each $X, Y \in \mathcal{D}$. By the Frobenius theorem, at each point $p \in M$, $\mathcal{D}$ then spans the tangent space to an *integral submanifold*, which has the same dimension as $\mathcal{D}$, and we then say $\mathcal{D}$ is *Frobenius integrable* provided this dimension is constant. The *kernel* or *characteristic space* of a differential form Ω is the span of vector fields which annihilate Ω, $\ker\Omega = \{X \in \chi(M) : X \lrcorner \Omega = 0\}$. A p-form Ω is integrable if its kernel is Frobenius integrable and of maximal dimension everywhere. Suppose that we have a Frobenius integrable distribution on E, $\mathcal{D} = \mathrm{span}(\{\Gamma, X_1, \ldots, X_{2n-1}\})$, of dimension $2n$ and a vector field $Z \in \chi(E)$ which is a symmetry of $\mathcal{D}$. Then the form

$$\omega := \frac{X_1 \lrcorner \ldots \lrcorner X_{2n-1} \lrcorner \Omega}{\Omega(X_1 \lrcorner \ldots \lrcorner X_{2n-1} \lrcorner Z)}$$

where $\Omega := \theta^1 \wedge \ldots \wedge \theta^n \wedge \phi^1 \ldots \wedge \phi^n$, is a linear combination of the contact and force forms (since $\Gamma \lrcorner \Omega = 0$). Further, ω is closed and so locally $\omega = \mathrm{d}f$ for some first integral f of Γ. Once we realize that $X_1 \lrcorner \ldots \lrcorner X_{2n-1} \lrcorner \Omega$ is a Frobenius integrable 1-form with kernel $\mathcal{D}$ with symmetry Z, the proof of this result is essentially a corollary to the following theorem.

Theorem. Let θ be a Frobenius integrable 1-form which is nowhere zero on some open subset U of a manifold N, then $\mathrm{d}(I\theta) = 0$ if and only if $I = (X \lrcorner \theta)^{-1}$ for some symmetry vector field X of θ on U (that is, $\mathcal{L}_X \theta = \beta\theta$, β a smooth function on M, or equivalently $\mathcal{L}_X \ker\theta = \ker\theta$) with $X \lrcorner \theta \neq 0$ on U.

The first result can be generalized to the following theorem.

Theorem. Let Ω be a k-form on a manifold N, and let $\mathrm{span}(\{X_1, \ldots, X_k\})$ be a k-dimensional distribution on open $U \subseteq N$ satisfying $X_i \lrcorner \Omega \neq 0$ everywhere on U. Further suppose that $\mathrm{span}(\{X_{j+1}, \ldots, X_k\} \cup \ker\Omega)$ is integrable for some $j < k$ and that X_i is a symmetry of $\mathrm{span}(\{X_{i+1}, \ldots, X_k\} \cup \ker\Omega)$ for $i = 1, \ldots, j$. Put

$$\sigma^1 = X_2 \lrcorner X_3 \lrcorner \ldots \lrcorner X_k \lrcorner \Omega$$

$$\sigma^2 = X_1 \lrcorner X_3 \lrcorner \ldots \lrcorner X_k \lrcorner \Omega$$

$$\vdots$$

$$\sigma^k = X_1 \lrcorner X_2 \lrcorner \ldots \lrcorner X_{k-1} \lrcorner \Omega$$

and

$$\omega^i = \frac{\sigma^i}{X_i \lrcorner \sigma^i} \qquad \text{for } i = 1, \ldots, k$$

so that $\{\omega^1, \ldots, \omega^k\}$ is dual to $\{X_1, \ldots, X_k\}$. Then $\mathrm{d}\omega^1 = 0$, $\mathrm{d}\omega^2 = 0 \bmod \omega^1$; $\mathrm{d}\omega^3 = 0 \bmod \omega^1, \omega^2; \ldots; \mathrm{d}\omega^j = 0 \bmod \omega^1, \ldots, \omega^{j-1}$, so that locally

$$
\begin{aligned}
\omega^1 &= \mathrm{d}\gamma^1 \\
\omega^2 &= \mathrm{d}\gamma^2 - X_1(\gamma^2)\mathrm{d}\gamma^1 \\
\omega^3 &= \mathrm{d}\gamma^3 - X_2(\gamma^3)\mathrm{d}\gamma^2 - \left(X_1(\gamma^3) - X_2(\gamma^3)X_1(\gamma^2)\right)\mathrm{d}\gamma^1 \\
&\ \vdots \\
\omega^j &= \mathrm{d}\gamma^j \bmod \mathrm{d}\gamma^1, \ldots, \mathrm{d}\gamma^{k-1}
\end{aligned}
$$

for some $\gamma^1, \ldots, \gamma^j \in \bigwedge^0 T^*U$ (functions or 0-forms on T^*U). Also the system $\{\omega^{j+1}, \ldots, \omega^k\}$ is integrable modulo $\mathrm{d}\gamma^1, \ldots, \mathrm{d}\gamma^j$ and locally $\Omega = \gamma^0 \mathrm{d}\gamma^1 \wedge \mathrm{d}\gamma^2 \wedge \cdots \wedge \mathrm{d}\gamma^j \wedge \omega^{j+1} \wedge \cdots \wedge \omega^k$ for some $\gamma^0 \in \bigwedge^0 T^*U$. Each γ^i is uniquely defined up to the addition of an arbitrary function of $\gamma^1, \ldots, \gamma^{i-1}$.

Using this theorem and our four vector fields $X_4^{(1)}, X_3^{(1)}, X_2^{(1)}$ and X_1, we can calculate four first integrals for the system. We first obtain four 1-forms:

$$
\begin{aligned}
\omega^1 &= \frac{X_2^{(1)} \lrcorner X_3^{(1)} \lrcorner X_4^{(1)} \lrcorner \Omega}{\Omega(X_4^{(1)}, X_3^{(1)}, X_2^{(1)}, X_1)} \\
\omega^2 &= \frac{X_1 \lrcorner X_3^{(1)} \lrcorner X_4^{(1)} \lrcorner \Omega}{\Omega(X_4^{(1)}, X_3^{(1)}, X_1, X_2^{(1)})} \\
\omega^3 &= \frac{X_1 \lrcorner X_2^{(1)} \lrcorner X_4^{(1)} \lrcorner \Omega}{\Omega(X_4^{(1)}, X_2^{(1)}, X_1, X_3^{(1)})} \\
\omega^4 &= \frac{X_1 \lrcorner X_2^{(1)} \lrcorner X_3^{(1)} \lrcorner \Omega}{\Omega(X_3^{(1)}, X_2^{(1)}, X_1, X_4^{(1)})} .
\end{aligned}
$$

where Γ is the characteristic vector field of the 4-form Ω, defined by

$$
\Omega = (\mathrm{d}r - \dot{r}\,\mathrm{d}t) \wedge (\mathrm{d}\theta - \dot{\theta}\,\mathrm{d}t) \wedge [\mathrm{d}\dot{r} - (r\dot{\theta}^2 - \mu/r^2)\mathrm{d}t] \wedge \left[\mathrm{d}\dot{\theta} + \left(\frac{2\dot{r}\dot{\theta}}{r}\right)\mathrm{d}t\right] .
$$

Now X_1 is a symmetry of the integrable distribution $\mathcal{D} = \mathrm{span}(\{\Gamma, X_4^{(1)}, X_3^{(1)}, X_2^{(1)}\})$ and so ω^1 is closed. Similarly, $X_2^{(1)}$ is a symmetry of the integrable distibution $\mathcal{D}' = \mathrm{span}(\{\Gamma, X_4^{(1)}, X_3^{(1)}, X_1\})$ and so ω^2 is also closed. Thus we obtain two first integrals from ω^1 and ω^2 which turn out to be (up to a sign)

$$
f^1 := h^2\mathcal{E} \qquad f^2 := 3\log|h|
$$

respectively.

From now on we will consider only the component E^+ of E for which $\dot{\theta} > 0$. Instead of X_1 it might appear $\hat{X}_1 = h^2 X_1 = (1/\dot{r})(\partial/\partial\dot{r})$ does just as well as a symmetry of the system, producing the only notable change $[X_1, X_2^{(1)}] = -\frac{2}{3}\hat{X}_1$ and leaving ω^2

closed. However, our original task was to find a X_1 as a symmetry of $\mathcal{D}$, which will not be the case for $\hat{X}_1$ and, furthermore, ω^1 will no longer be closed. ω^3 and ω^4 are not closed, and we can still obtain two additional first integrals by $\omega^3 = \mathrm{d}f^3 \bmod \mathrm{d}f^1, \mathrm{d}f^2$ and $\omega^4 = \mathrm{d}f^4 \bmod \mathrm{d}f^1, \mathrm{d}f^2$ giving (up to a sign)

$$f^3 = \begin{cases} \dfrac{-r\dot{r}}{2\mathcal{E}} + \dfrac{\mu}{(-2\mathcal{E})^{3/2}} \sin^{-1}\left(\dfrac{2\mathcal{E}r+\mu}{R}\right) + t & -\mu^2/2h^2 < \mathcal{E} < 0 \\ \dfrac{-(\mu r + h^2) r\dot{r}}{3\mu^2} + t & \mathcal{E} \doteq 0 \\ \dfrac{-r\dot{r}}{2\mathcal{E}} + \dfrac{\mu}{(2\mathcal{E})^{3/2}} \log(2\sqrt{2\mathcal{E}}r\dot{r} + 4\mathcal{E}r + 2\mu) + t & \mathcal{E} > 0 \end{cases}$$

$$f^4 = \begin{cases} \sin^{-1}\left(\dfrac{\mu r - h^2}{rR}\right) - \theta & \mathcal{E} \neq 0 \\ 2\tan^{-1}\left(\dfrac{r^2\dot{r}^2}{h^2}\right) - \theta & \mathcal{E} = 0 \end{cases}$$

where $R^2 = \mu^2 + 2h^2\mathcal{E}$ is the square of the Runge–Lenz vector. f^3 and f^4 represent initial values of t and θ, respectively. We remark that doing these integrations is tantamount to explicitly solving the equations of motion $(1a)$, $(1b)$, while producing f^1 and f^2 is not.

We now have local bases of 1-forms for E^+, namely $\{\mathrm{d}t, \mathrm{d}f^1, \mathrm{d}f^2, \mathrm{d}f^3, \mathrm{d}f^4\}$ such that $\Gamma \lrcorner\, \mathrm{d}f^i = 0$ and $\Gamma \lrcorner\, \mathrm{d}t = 1$, for Γ defined in (2) and f^1, f^2, f^3, f^4 the conserved quantities of the system. We can calculate the dual coordinate basis vectors and construct local bases for $\chi(E^+)$, denoted $\{\Gamma, \partial/\partial f^1, \partial/\partial f^2, \partial/\partial f^3, \partial/\partial f^4\}$. With the aid of REDUCE and our expressions for $\omega^1, \omega^2, \omega^3, \omega^4$ we obtain the relations

$$X_4^{(1)} = \frac{\partial}{\partial f^3} + \Gamma$$

$$X_3^{(1)} = \frac{\partial}{\partial f^4}$$

$$X_2^{(1)} = \frac{\partial}{\partial f^1} + f^3 \frac{\partial}{\partial f^3} + t\Gamma$$

$$X_1 = \begin{cases} \dfrac{\partial}{\partial f^1} - \left[\dfrac{3\mu}{(-2\mathcal{E})^{5/2}h^2} \sin^{-1}\left(\dfrac{2\mathcal{E}r+\mu}{R}\right) + \dfrac{2r(\dot{r}^2+\mathcal{E})}{(-2\mathcal{E})^2h^2\dot{r}} \right. \\ \quad \left. - \dfrac{\mu(-2h^2r\mathcal{E} + h^2\mu - 2r\mu^2)}{(-2\mathcal{E})^2h^2r\dot{r}R}\right] \dfrac{\partial}{\partial f^3} + \left(\dfrac{\mu r - h^2}{hR^2r\dot{r}}\right) \dfrac{\partial}{\partial f^4} & -\mu^2/2h^2 < \mathcal{E} < 0 \\ \dfrac{\partial}{\partial f^1} - \left(\dfrac{(\mu r + h^2)r}{3\mu^2\dot{r}h^2}\right) \dfrac{\partial}{\partial f^3} + \left(\dfrac{4r^2}{(h^4 + r^4\dot{r}^4)}\right) \dfrac{\partial}{\partial f^4} & \mathcal{E} = 0 \\ \dfrac{\partial}{\partial f^1} + \left(\dfrac{3\mu \log(2(\sqrt{2\mathcal{E}}\dot{r}r + 2\mathcal{E}r + \mu))}{(2\mathcal{E})^{5/2}h^2} - \dfrac{2r}{(2\mathcal{E})^{3/2}h^2} \right. \\ \quad \left. - \dfrac{r(4\mathcal{E}^2 - 3\dot{r}^2\mu + 2\sqrt{2\mathcal{E}}\dot{r}^3 - \sqrt{2\mathcal{E}}\mathcal{E}r\dot{r})}{(2\mathcal{E})^2h^2\dot{r}(\sqrt{2\mathcal{E}}r\dot{r} + 2r\mathcal{E} + \mu)}\right) \dfrac{\partial}{\partial f^3} + \left(\dfrac{\mu r - h^2}{hR^2r\dot{r}}\right) \dfrac{\partial}{\partial f^4} & \mathcal{E} > 0. \end{cases}$$

At this point it is worth making a few remarks to identify the novel aspects of this technique. First of all, we do not require the vector fields that we use to be symmetries of Γ as, for example, Noether's theorem does. X_1 is a case in point: it is much easier to find than the symmetry $\partial/\partial f^1$ and it is just as effective in producing first integrals. Secondly, the fact that these fields may be interpreted as symmetries of a reduced differential equation in general frees us from the necessity of using canonical coordinates and quotients. We used them here because we wanted to demonstrate that canonical coordinates are still useful and because we still need some quotient ideas to establish the periodicity of the negative energy orbits in the last section. But they are by no means the only way to find these vector fields. The third point is that first integrals are produced in the original coordinate chart from a very simple formula, a great advantage in algebraic computing. Indeed, the second theorem of this section shows us exactly which vector fields to look for to minimize the number of quadratures in any given problem.

4. Closure of $\mathcal{E} < 0$ orbits

We will now demonstrate that the negative energy orbits are periodic in time using only the first integral $h^2\mathcal{E}$. We begin by showing that the integral curves of Γ on E^+ for which $h^2\mathcal{E} < 0$ project to closed curves on the quotient by the action of $X_4^{(1)}, X_3^{(1)}, X_2^{(1)}$. We then use this result to prove that orbits on M for which $h^2\mathcal{E} < 0$ are closed and furthermore are periodic in time.

On passing to the quotient of E^+ by the action of $X_4^{(1)}, X_3^{(1)}, X_2^{(1)}$ we found that the integral curves of Γ project to curves with tangent

$$\hat{\Gamma} := \left(v^1 + 2\frac{v^{2^2}}{v^1} - 2\frac{\mu}{v^1}\right)\frac{\partial}{\partial v^1} - v^2\frac{\partial}{\partial v^2}.$$

By the first theorem of the previous section, if we have a 1-form ω on the quotient (here Frobenius integrable by dimension) such that $\hat{\Gamma} \lrcorner\, \omega = 0$, and a symmetry of $\hat{\Gamma}$ (here we have X_1) we obtain a closed 1-form

$$\mathrm{d}g = \frac{\omega}{(X_1 \lrcorner\, \omega)}$$

where g is a conserved function of $\hat{\Gamma}$, i.e. $\hat{\Gamma}(g) = 0$. By substituting

$$X_1 = \frac{1}{v^1 v^{2^2}}\frac{\partial}{\partial v^1}$$

$$\omega = v^2\,\mathrm{d}v^1 + \left(v^1 + 2\frac{v^{2^2}}{v^1} - 2\frac{\mu}{v^1}\right)\mathrm{d}v^2$$

we obtain

$$\mathrm{d}g = v^1 v^2\left[v^2\,\mathrm{d}v^1 + \left(v^1 + 2\frac{v^{2^2}}{v^1} - 2\frac{\mu}{v^1}\right)\mathrm{d}v^2\right].$$

Solving for g gives

$$g = \tfrac{1}{2}v^{2^2}(v^{1^2} + v^{2^2} - 2\mu).$$

By substituting $v^1 = \dot{r}r^{1/2}$ and $v^2 = \dot{\theta}r^{3/2}$ we see that, by an obvious abuse of notation,

$$\begin{aligned} g &= h^2[\tfrac{1}{2}(\dot{r}^2 + h^2/r^2) - \mu/r] \\ &= h^2\mathcal{E}. \end{aligned}$$

Alternatively, it follows from the previous expression for $\omega^1 = \mathrm{d}(h^2\mathcal{E})$ that $h^2\mathcal{E}$ lives on these submanifolds with $\hat{\Gamma}(h^2\mathcal{E}) = 0$ and $X_1(h^2\mathcal{E}) = 1$.

Consequently, the integral curves of $\hat{\Gamma}$ can be identified with the curves on E^+, $h^2\mathcal{E} =$ constant, $t =$ constant, $\theta =$ constant, $r^* =$ constant. Geometrically, at each point $(t, r^*, \theta, v^1, v^2)$ of E^+ there is an integral curve of Γ with a fixed value of $h^2\mathcal{E}$, K say, and an integral curve of $\hat{\Gamma}$ with $h^2\mathcal{E} = K$; the collection of these latter curves along the entire integral curve of Γ can loosely be identified with the projection of the integral curve to the quotient. We wish to investigate the properties of these projections for all values of $\mathcal{E}$. Observing that g is a function of v^{1^2} and v^{2^2}, $g(v^{1^2}, v^{2^2}) = K$, where K is a non-zero constant, defines a curve on a $v^2 > 0$ surface which is symmetric about the v^2 axis. This curve will be closed if it is continuous and crosses the v^2 axis twice. We want to determine the values of K for which the curves are closed. Setting $h^2\mathcal{E} = K$ gives (after some rearranging) $v^{1^2} = 2\mu - v^{2^2} + 2K/v^{2^2}$ ($v^2 > 0$ by assumption). Setting $v^1 = 0$ and rearranging the terms, we get the quartic in v^2, $v^{2^4} - 2\mu v^{2^2} - 2K = 0$, which has roots $v^{2^2} = \mu \pm \sqrt{\mu^2 + 2K}$. To obtain two values we require $0 < \mu^2 + 2K < \mu^2$, i.e. $-\mu^2 < 2K < 0$. Since $K = h^2\mathcal{E}$, this means $-\mu^2/2h^2 < \mathcal{E} < 0$. Hence the curve cuts the v^2 axis at $(\mu + \sqrt{\mu^2 + 2h^2\mathcal{E}})^{1/2}$ and $(\mu - \sqrt{\mu^2 + 2h^2\mathcal{E}})^{1/2}$ and is thus closed for this range of $\mathcal{E}$ (continuity is self-evident). For $2K > 0$ we have only one value for v^{2^2} since $v^{2^2} > 0$, namely $v^{2^2} = \mu + \sqrt{\mu^2 + 2K}$. Therefore, the curve will only cut the v^2 axis once at $(\mu + \sqrt{\mu^2 + 2K})^{1/2}$ and hence for $K > 0$, i.e. for $\mathcal{E} > 0$ these curves are not closed. The intermediate case $K = 0$ corresponds to the open curves $v^2 = 0$ (radial orbits excluded from E^+) or the open curves $v^{1^2} + v^{2^2} = 2\mu$, $v^2 \neq 0$. (The entire argument works for the case $v^2 < 0$ and this corresponds to the time reversal symmetry in the problem.)

We have, in effect, shown that the projection of any integral curve of Γ with $h^2\mathcal{E} < 0$ onto the quotient is closed, thus $h^2\mathcal{E} < 0$ is a necessary condition for the orbits on the base to be closed. Now we will show that this is also a sufficient condition without having to solve the equations of motion. If TM^+ denotes the component of TM for which $\dot{\theta} > 0$ then the projection of Γ by the action of $\partial/\partial t$ can be identified with the field

$$\Gamma^* := \frac{3}{2}v^1\frac{\partial}{\partial r^*} + v^2\frac{\partial}{\partial \theta} + \left(\frac{1}{2}v^{1^2} + v^{2^2} - \mu\right)\frac{\partial}{\partial v^1} - \frac{1}{2}v^1v^2\frac{\partial}{\partial v^2}$$

(up to a reparametrization) on TM^+ in the chart (r^*, θ, v^1, v^2). Now the projection of Γ^* to the quotient is tangent to closed curves for $h^2\mathcal{E} < 0$ as we have seen. On any one of these closed curves, we choose a global parametrization $\alpha : R \to M, (v^1, v^2)_p = (\alpha^1(s), \alpha^2(s))$ with $\Gamma^*(s) = 1$; this is periodic because the curve has a unique tangent vector field $\dot\alpha$ (the components of Γ^* are independent of s). Hence for some s_1, s_2, $\alpha(s_1) = \alpha(s_2)$ and the two reparametrizations, $\alpha^*(s) := \alpha(s+s_1)$ and $\alpha^\dagger(s) := \alpha(s+s_2)$ satisfy $\alpha^*(0) = \alpha^\dagger(0)$ and $\dot\alpha^*(s) = \dot\alpha^\dagger(s)$, so by the uniqueness of integral curves of this tangent vector field, $\alpha^*(s) = \alpha^\dagger(s)$ $(= \alpha(s))$ for all s. That is, $\alpha(s+s_1) = \alpha(s+s_2)$ for all s, and hence $\alpha(s) = \alpha(s+(s_2-s_1))$ for all s so that the curve is periodic with period T less than or equal to $(s_2 - s_1)$.

The remaining differential equations for the $h^2\mathcal{E} < 0$ integral curve of Γ^* through $p \in TM^+$ are

$$\frac{\mathrm{d}r^*}{\mathrm{d}s} = \frac{3}{2}\alpha^1(s) \qquad \frac{\mathrm{d}\theta}{\mathrm{d}s} = \alpha^2(s)$$

with some appropriate initial conditions $r^*(0) = r_0^*$ and $\theta(0) = \theta_0$. Rather than solve these equations we will use the features of the curves on the quotient to prove that this integral curve of Γ^* is closed.

Using the expression for v^1 as a function of v^2 from the previous analysis and the relation $(\mathrm{d}v^2 + \frac{1}{2}v^1\mathrm{d}\theta)(\Gamma^*) = 0$, we find that $\sin(\theta_0 - \theta(s)) = (\mu - \alpha^{2^2}(s))/(\mu^2+2K)^{1/2}$ (recall that $\mu^2 + 2K > 0$), and since α^2 is periodic in s with period T we have $\theta(s+T) - \theta(s) = 2\pi$. That is, θ increases by 2π on each traversal of the closed curve on the quotient.

In addition, r^* is a periodic function of s with period T because the closed curves are symmetric about the v^2 axis:

$$\begin{aligned} r^*(p) &= \frac{3}{2}\int_0^s \alpha^1(s)\,\mathrm{d}s + r_0^* \\ &= \frac{3}{2}\int_0^{s-kT} \alpha^1(s)\,\mathrm{d}s + r_0^* \end{aligned}$$

where kT is the largest integer multiple of T less than s. (The last equality comes from $\int_0^T \alpha^1(s)\,\mathrm{d}s = 0$.) So it is clear that on this integral curve of Γ^* the coordinate r^* takes the same value at points where the angular coordinate is $\theta(p)$ and $\theta(p) + 2k\pi$. Hence this curve is closed (the tangents at both values of θ are the same) and so r^* and $r = \frac{2}{3}\exp(r^*)$ are periodic functions of θ. It is also clear that the corresponding orbit $(r(s), \theta(s))$ on M is closed and hence $h^2\mathcal{E} < 0$ is a sufficient condition for orbit closure. (We could have used the θ coordinate along the integral curves of Γ^* to parametrize the projected closed curves, but we needed at least some of the above reasoning to show that θ takes all values.)

The periodicity in time of the negative energy orbits is a straightforward deduction from their closure and time translation invariance: the argument is precisely the one we used above to deduce that the α parametrization was periodic. Similarly, the negative energy orbits for $\dot\theta < 0$ are also closed. The period $P = 2\pi\mu(-2\mathcal{E})^{-3/2}$ is easily deduced from the expression for f^3 now that we know these orbits are periodic. However, this last deduction effectively relies on the solution of the equations of motion while the periodicity argument does not.

5. Conclusion

In this paper we have found a symmetry of the integrable distribution of Γ and its symmetries, $\mathcal{D} = \mathrm{span}(\{\Gamma, X_4^{(1)}, X_3^{(1)}, X_2^{(1)}\})$ for the Kepler problem. Using the four vector fields $X_4^{(1)}$, $X_3^{(1)}$, $X_2^{(1)}$ and X_1, we obtained four conserved quantites for the system in a relatively straightforward manner. We further reproduced the well-known result that, for $-\mu^2/2h^2 < \mathcal{E} < 0$, the solution curves are closed and periodic. This was achieved by reducing the second-order differential equation vector field down to a two-dimensional first-order differential equation field and obtaining a first integral of the projected vector field, namely $h^2\mathcal{E}$. The integral curves of this field are closed for the above range of E. The closure of the negative energy orbits of our original second-order field folllows without having to produce any further first integrals other than h (to calculate the period of the θ parametrization), an integral we needed anyway to reduce the motion to a plane. The periodicity of these orbits is essentially due to the time translation invariance of the problem. Our approach required only symmetries and the integration of a closed 1-form and certainly did not need anything equivalent to a full solution technique for investigating the constant energy submanifolds. Our approach should be applicable to any first integral, and we hope to use the method to probe the existence of closed orbits for other problems where explicit solutions are difficult to obtain.

Finally, we trust that this paper will serve to popularize the use of a manifestly geometric approach to reduction of order which originated with the work of E Cartan (we refer the reader to [1] for a fairly full account of the ideas we have used).

Acknowledgments

We both thank James Sherring, Grant Cairns, Norm Dancer, Chris Radford and a referee for their help. S E Godfrey acknowledges an Australian Postgraduate Research Award and the hospitality of the Department of Mathematics, Statistics and Computing Science at the University of New England, Armidale.

References

[1] Sherring J and Prince G 1990 *Geometric Aspects of Reduction of Order* Department of Mathematics, Statistics and Computing Science Research Report 90 – 20, University of New England, Armidale, NSW, 2351

[2] Prince G E and Eliezer C J 1981 *J. Phys. A: Math. Gen.* **14** 587–96

[3] Whittaker E T 1927 *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge: Cambridge University Press)

[4] Landau L D and Lifshitz E M 1976 *Mechanics* 3rd edn (Oxford: Pergamon)

[5] Wulfman C E and Wybourne B G 1976 *J. Phys. A: Math. Gen.* 9 507–18

[6] Prince G E and Crampin M 1984 *Gen. Rel. Grav.* 16 921–41

[7] Olver P 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)